Ancestral Logic: a Proof Theoretical Study

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Abstract. Many efforts have been made in recent years to construct formal systems for mechanizing mathematical reasoning. A framework which seems particularly suitable for this task is *ancestral logic* – the logic obtained by augmenting first-order logic with a transitive closure operator. While the study of this logic has so far been mostly model-theoretical, this work is devoted to its proof theory (which is much more relevant for the task of mechanizing mathematics). We develop a Gentzen-style proof system TC_G which is sound for ancestral logic, and prove its equivalence to previous systems for the reflexive transitive closure operator by providing translation algorithms between them. We further provide evidence that TC_G indeed encompasses all forms of reasoning for this logic that are used in practice. The central rule of TC_G is an induction rule which generalizes that of Peano Arithmetic (PA). In the case of arithmetics we show that the ordinal number of TC_G is ε_0.

Keywords: ancestral logic, transitive closure, proof theory, Gentzen-style systems, constructive consistency proofs.

1 Introductions

In light of recent advances in the field of automated reasoning, formal systems for mechanizing mathematical reasoning are attracting a lot of interest (see, e.g., [11,5,6,15]). Most of these systems go beyond first-order logic (FOL), because the latter is too weak for this task: one cannot even give in it a categorical characterization of the most basic concept of mathematics - the natural numbers. Using second-order logic (SOL) for this task, however, has many disadvantages. SOL has doubtful semantics, as it is based on debatable ontological commitments. Moreover, it does not seem satisfactory that dealing with basic notions (such as the natural numbers) requires using the strong notions involved in SOL, such as quantifying over all subsets of infinite sets. In addition, SOL is difficult to deal with from a proof-theoretical point of view.

The above considerations imply that the most suitable framework for mechanizing mathematical reasoning should be provided by some logic between FOL and SOL. A framework that seems particularly suitable for this task is *ancestral logic* – the logic obtained by augmenting FOL with the concept of transitive closure of a given relation. Indeed, ancestral logic provides a suitable framework for the formalization of mathematics as it is appropriate for defining fundamental abstract formulations of transitive relations that occur commonly in basic mathematics (see, e.g., [2,16,17]).

Most of the works on ancestral logic have so far been carried out in the context of finite model theory (see, e.g., [7]). Clearly, the focus on finite structures renders these works irrelevant for the task of formalizing mathematics. Moreover, most of this research has been dedicated to model theory, while for mechanizing mathematics we need useful *proof systems*.

This work provides a proof-theoretical study of ancestral logic. In [2] a formal proof system for ancestral logic was suggested. Therein it was stated that: "a major research task here is to find out what other rules (if any) should be added in order to make the system 'complete' in some reasonable sense". In this work we provide an answer to this question. We show that the system proposed in [2] is too weak, as it fails to prove certain fundamental properties of the transitive closure operator. We then take further steps towards a useful proof system for ancestral logic by proposing a stronger system, TC_G, which is sound for this logic and encompasses all forms of reasoning for this logic that are used in practice. TC_G is proven to be equivalent to systems previously suggested in the literature for the reflexive transitive closure, in the sense that there are translation algorithms between them that preserve provability. We further investigate the proof theoretical method of constructive consistency proofs and show that in the case of arithmetics the ordinal number of the system TC_G is ε_0.

Due to lack of space, most proofs in the paper are omitted, and will appear in an extended version. An appendix with full proofs is enclosed in this submitted draft, to be consulted at the discretion of program committee members.

2 Logics with a Transitive Closure Operator

In mathematics, the transitive closure of a binary relation R is defined as the minimal transitive relation that contains R. In general, the transitive closure operator, TC, is not first-order definable (see, e.g., [8,1]). Thus, we present ancestral logic, which is the logic obtained by augmenting FOL with a transitive closure operator[1]. Below are the corresponding formal definitions of a first-order language augmented by a transitive closure operator, and its semantics.

In this paper σ denotes a first-order signature with equality. A structure for a first-order language based on σ is an ordered pair $M = \langle D, I \rangle$, where D is a non-empty set of elements (the domain) and I is an interpretation function on σ. To avoid confusion regarding parentheses, we use (,) for parentheses in a formal language, and [,] for parentheses in the metalanguage.

Definition 1. *Let σ be a signature for a first-order language with equality, and let $M = \langle D, I \rangle$ be a structure for σ and v an assignment in M.*

- *The language $L_{TC}(\sigma)$ is defined as the first-order language based on σ, with the addition of the TC operator defined by: for any formula φ in $L_{TC}(\sigma)$, x, y distinct variables, and s, t terms, $(TC_{x,y}\varphi)(s,t)$ is a formula in $L_{TC}(\sigma)$. The free occurrences of x and y in φ are bound in this formula.*

[1] Such logics are also sometimes called Transitive Closure Logic.

- *The pair $\langle M, v\rangle$ is said to satisfy $(TC_{x,y}\varphi)(s,t)$ if there exist $a_0, ..., a_n \in D$ ($n > 0$) such that $v[s] = a_0$, $v[t] = a_n$, and φ is satisfied by M and $v[x := a_i,\ y := a_{i+1}]$[2] for $0 \leq i \leq n-1$.*
  The logic obtained is called Ancestral Logic and it is denoted by $\mathcal{L}_{TC}$.

In the semantics presented here, $(TC_{x,y}\varphi)(s,t)$ requires that there should be at least one φ-step between s and t. However, another well studied form of the transitive closure operator [12,13,14] is the reflexive form, RTC.

Definition 2. *Let σ be a first-order signature, and let $M = \langle D, I\rangle$ be a structure for σ and v an assignment in M.*

- *The language $L_{RTC}(\sigma)$ is defined as $L_{TC}(\sigma)$ with TC replaced by RTC.*
- *The pair $\langle M, v\rangle$ is said to* satisfy *$(RTC_{x,y}\varphi)(s,t)$ if $s = t$ or there exist $a_o, ..., a_n \in D$ ($n > 0$) such that $v[s] = a_0$, $v[t] = a_n$, and φ is satisfied by M and $v[x := a_i,\ y := a_{i+1}]$ for $0 \leq i \leq n-1$.*
  Similarly, the obtained logic is denoted by $\mathcal{L}_{RTC}$.

Using equality, the two forms of the transitive closure operator are definable in terms of each other. The reflexive transitive closure operator is definable using the non-reflexive form by

$$(RTC_{x,y}\varphi)(s,t) := (TC_{x,y}\varphi)(s,t) \vee s = t,$$

while the non-reflexive TC operator is definable, for example, by

$$(TC_{x,y}\varphi)(s,t) :\!= \exists z \left(\varphi\left\{\frac{s}{x}, \frac{z}{y}\right\} \wedge (RTC_{x,y}\varphi)(z,t)\right)$$

where z is a fresh variable.[3]

One difference between the two forms is the ability to define quantifiers. The existential quantifier can be defined using the TC operator [2], however it cannot be defined using the RTC operator, as we prove below.

Proposition 1. *The existential quantifier is not definable in the quantifier-free fragment of $\mathcal{L}_{RTC}$.*

Proof. Take σ to consist of a constant symbol 0 and a unary predicate symbol P. It can be easily shown by induction that each quantifier-free sentence ψ in $\mathcal{L}^{\sigma}_{RTC}$ is logically equivalent to one of the following sentences: $P(0)$, $\neg P(0)$, $0 = 0$, or $0 \neq 0$. Since $\exists x P(x)$ is clearly not logically equivalent to any of these four sentences, we conclude that the existential quantifier cannot be defined in the quantifier-free fragment of $\mathcal{L}_{RTC}$. $\square$

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[2] $v[x := a]$ denotes the x-variant of v which assigns to x the element a from D.

[3] $\varphi\left\{\frac{t_1}{x_1}, ..., \frac{t_n}{x_n}\right\}$ denotes the formula obtained from φ by substituting t_i for each free occurrence of x_i in φ, assuming that $t_1, ..., t_n$ are free for $x_1, ...x_n$ in φ.

The concept of the transitive closure operator is embedded in our understanding of the natural numbers. Therefore, it is only natural to explore the expressive power of various first-order languages for arithmetic augmented by the TC operator. Let 0 be a constant symbol and s a unary function symbol. It is known that in $\mathcal{L}_{TC}^{\{0,s\}}$ together with the standard axioms for the successor function, the following sentence categorically characterize the natural numbers:

$$\forall x \left(x = 0 \vee \left(TC_{w,u} \left(s(w) = u \right) \right) (0, x) \right) \tag{1}$$

In [2] it was also shown that all recursive functions and relations are definable in $\mathcal{L}_{TC}^{\{0,s,+\}}$, where $+$ is a binary function symbol. This implies that the upward Löwenheim-Skolem theorem fails for ancestral logic, and that ancestral logic is finitary, i.e. the compactness theorem fails for it. Moreover, ancestral logic is not even arithmetic, thus any formal deductive system which is sound for it is incomplete.

3 Gentzen-Style Proof Systems for Ancestral Logic

Ideally, we would like to have a consistent, sound, and complete axiomatic system for ancestral logic. However, since there could be no sound and complete system for ancestral logic, one should instead look for useful and effective *partial* formal systems that are still adequate for formalizing mathematical reasoning. The systems defined in this section are extensions of Gentzen's system for classical first-order logic with equality, $\mathcal{LK}_=$ [9].

In what follows the letters Γ, Δ represent finite (possibly empty) multisets of formulas, φ, ψ, ϕ arbitrary formulas, x, y, z, u, v, w variables, and r, s, t terms. For convenience, we shall denote a sequent of the form $\Gamma \Rightarrow \{\varphi\}$ by $\Gamma \Rightarrow \varphi$, and employ other standard abbreviations, such as Γ, Δ instead of $\Gamma \cup \Delta$. To improve readability, in some derivations we omit the context from the sequents.

In [12,13,14] two equivalent Hilbert-style systems for ancestral logic in which the reflexive transitive closure operator, RTC, was taken as primitive were suggested. Below is a Gentzen-style proof system for the RTC operator which is equivalent to the Hilbert-style systems presented in the original papers.

Definition 3.
The system RTC_G is defined by adding to $\mathcal{LK}_=$ the axiom

$$\Gamma \Rightarrow \Delta, \left(RTC_{x,y} \varphi \right) (s, s) \tag{2}$$

and the following inference rules:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \left\{ \frac{s}{x}, \frac{t}{y} \right\}}{\Gamma \Rightarrow \Delta, \left(RTC_{x,y} \varphi \right) (s, t)} \tag{3}$$

$$\frac{\Gamma \Rightarrow \Delta, \left(RTC_{x,y} \varphi \right) (s, r) \quad \Gamma \Rightarrow \Delta, \left(RTC_{x,y} \varphi \right) (r, t)}{\Gamma \Rightarrow \Delta, \left(RTC_{x,y} \varphi \right) (s, t)} \tag{4}$$

$$\frac{\Gamma, \psi(x), \varphi(x,y) \Rightarrow \Delta, \psi\left\{\frac{y}{x}\right\}}{\Gamma, \psi\left\{\frac{s}{x}\right\}, (RTC_{x,y}\varphi)(s,t) \Rightarrow \Delta, \psi\left\{\frac{t}{x}\right\}} \tag{5}$$

In all three rules we assume that the terms which are substituted are free for substitution and that no forbidden capturing occurs. In Rule (5) x should not occur free in Γ and Δ, and y should not occur free in Γ, Δ and ψ.

Rule (5) is a generalized induction principle which states that if t is a φ-descendant of s (or equal to it), then if s has some property which is passed down from one object to another if they are φ-related, then t also has that property.

We next show that RTC_G is adequate for RTC, in the sense that it does give the RTC operator the intended meaning of the reflexive transitive closure, and can derive all fundamental rules concerning the RTC operator that have been suggested in the literature (as far as we know).

Proposition 2. *The following rules are derivable in RTC_G[4]:*

$$\frac{\Gamma \Rightarrow \Delta, \varphi\left\{\frac{s}{x}, \frac{r}{y}\right\} \quad \Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(r,t)}{\Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(s,t)} \qquad \frac{\Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(s,r) \quad \Gamma \Rightarrow \Delta, \varphi\left\{\frac{r}{x}, \frac{t}{y}\right\}}{\Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(s,t)} \tag{6}$$

$$\frac{\Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(s,t)}{\Gamma \Rightarrow \Delta, s=t, \exists z\left((RTC_{x,y}\varphi)(s,z) \wedge \varphi\left\{\frac{z}{x}, \frac{t}{y}\right\}\right)} \qquad \frac{\Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(s,t)}{\Gamma \Rightarrow \Delta, s=t, \exists z\left(\varphi\left\{\frac{s}{x}, \frac{z}{y}\right\} \wedge (RTC_{x,y}\varphi)(z,t)\right)} \tag{7}$$

$$\frac{\Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(s,t)}{\Gamma \Rightarrow \Delta, (RTC_{y,x}\varphi)(t,s)} \qquad \frac{(RTC_{x,y}\varphi)(s,t), \Gamma \Rightarrow \Delta}{(RTC_{y,x}\varphi)(t,s), \Gamma \Rightarrow \Delta} \tag{8}$$

$$\frac{\Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(s,t)}{\Gamma \Rightarrow \Delta, \left(RTC_{u,v}\varphi\left\{\frac{u}{x}, \frac{v}{y}\right\}\right)(s,t)} \qquad \frac{(RTC_{x,y}\varphi)(s,t), \Gamma \Rightarrow \Delta}{\left(RTC_{u,v}\varphi\left\{\frac{u}{x}, \frac{v}{y}\right\}\right)(s,t), \Gamma \Rightarrow \Delta} \tag{9}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma, (RTC_{x,y}\varphi)(s,t) \Rightarrow \Delta, (RTC_{x,y}\psi)(s,t)} \tag{10}$$

$$\frac{(RTC_{x,y}\varphi)(s,t), \Gamma \Rightarrow \Delta}{(RTC_{u,v}(RTC_{x,y}\varphi)(u,v))(s,t), \Gamma \Rightarrow \Delta} \tag{11}$$

$$\frac{\varphi\left\{\frac{s}{x}\right\}, \Gamma \Rightarrow \Delta}{(RTC_{x,y}\varphi)(s,t), \Gamma \Rightarrow s=t, \Delta} \qquad \frac{\varphi\left\{\frac{t}{y}\right\}, \Gamma \Rightarrow \Delta}{(RTC_{x,y}\varphi)(s,t), \Gamma \Rightarrow s=t, \Delta} \tag{12}$$

[4] These rules are counterparts of the Hilbert-style rules suggested in [12,13,14].

Conditions:

- In all the rules we assume that the terms which are substituted are free for substitution and that no forbidden capturing occurs.
- In (7) z should not occur free in Γ, Δ and $\varphi\left\{\frac{s}{x}, \frac{t}{y}\right\}$.
- In (9) the conditions are the usual ones concerning the α-rule.
- In (10) x, y should not occur free in Γ, Δ.
- In (11) u, v should not occur free in φ.
- In (12) y should not occur free in Γ, Δ or s in the left rule, and x should not occur free in Γ, Δ or t in the right rule.

In [2] a Gentzen-style system for the non-reflexive transitive closure operator was presented. Therein it was stated that: "a major research task here is to find out what other rules (if any) should be added in order to make the system 'complete' in some reasonable sense". In this section we answer this (two part) research question. First we show that the system in [2] is too weak for ancestral logic, as it fails to prove certain fundamental properties of the transitive closure operator. Then we present a stronger variation of the system which encompasses all forms of reasoning for ancestral logic that are used in practice.

Below is the proof system for the TC operator suggested in [2].

Definition 4.
The system TC'_G is defined by adding to $\mathcal{LK}_=$ the following inference rules:

$$\frac{\Gamma \Rightarrow \Delta, \varphi\left\{\frac{s}{x}, \frac{t}{y}\right\}}{\Gamma \Rightarrow \Delta, \left(TC_{x,y}\varphi\right)(s,t)} \tag{13}$$

$$\frac{\Gamma \Rightarrow \Delta, \left(TC_{x,y}\varphi\right)(s,r) \quad \Gamma \Rightarrow \Delta, \left(TC_{x,y}\varphi\right)(r,t)}{\Gamma \Rightarrow \Delta, \left(TC_{x,y}\varphi\right)(s,t)} \tag{14}$$

$$\frac{\Gamma, \psi(x), \varphi(x,y) \Rightarrow \Delta, \psi\left\{\frac{y}{x}\right\}}{\Gamma, \psi\left\{\frac{s}{x}\right\}, \left(TC_{x,y}\varphi\right)(s,t) \Rightarrow \Delta, \psi\left\{\frac{t}{x}\right\}} \tag{15}$$

The same restrictions on the rules in RTC_G apply here.

While all fundamental rules concerning RTC that have been suggested in the literature (as far as we know) are derivable in RTC_G, as shown in Prop. 2, in TC'_G this is not the case. There are fundamental properties of the TC operator which are unprovable in TC'_G.

Proposition 3. *The following valid sequents are unprovable in TC'_G:*

$$\begin{aligned}
&\left(TC_{x,y}\varphi\right)(s,t) \Rightarrow \varphi\left\{\frac{s}{x}, \frac{t}{y}\right\}, \exists z\left(\left(TC_{x,y}\varphi\right)(s,z) \wedge \varphi\left\{\frac{z}{x}, \frac{t}{y}\right\}\right)\\
&\left(TC_{x,y}\varphi\right)(s,t) \Rightarrow \varphi\left\{\frac{s}{x}, \frac{t}{y}\right\}, \exists z\left(\varphi\left\{\frac{s}{x}, \frac{z}{y}\right\} \wedge \left(TC_{x,y}\varphi\right)(z,t)\right)
\end{aligned} \tag{16}$$

$$(TC_{x,y}\varphi)(s,t) \Rightarrow \varphi\left\{\frac{s}{x}\right\} \qquad (TC_{x,y}\varphi)(s,t) \Rightarrow \varphi\left\{\frac{t}{y}\right\} \tag{17}$$

where in (16) z is a fresh variable and in (17) y does not occur free in $\varphi\left\{\frac{s}{x}\right\}$ in the left sequent, and x does not occur free in $\varphi\left\{\frac{t}{y}\right\}$ in the right sequent.

Proof. Suppose the above sequents are derivable in TC'_G. It is easy to see that all the rules in TC'_G remain valid and derivable in RTC_G if we replace the operator TC with RTC. Hence, the corresponding sequents for RTC are provable in RTC_G. However, they are obviously not valid, since $(RTC_{x,y}\varphi)(s,s)$ holds for all s and φ. □

In general, any sequent which is valid only for the TC operator and not for the RTC operator will not be derivable in TC'_G. The next natural question is how should the system TC'_G be altered in order to be able to derive in it all the basic rules for the TC operator that are used in practice. Recall that one of the mathematical definitions of the transitive closure of a relation R is the least transitive relation that contains R. Hence, we generalize TC'_G's induction rule in a way that correlates with the minimality requirement in the definition.

Definition 5.
The system TC_G is obtained from TC'_G by replacing Rule (15) by:

$$\frac{\Gamma, \varphi(x,y) \Rightarrow \Delta, \phi(x,y) \quad \Gamma, \phi\left\{\frac{u}{x}, \frac{v}{y}\right\}, \phi\left\{\frac{v}{x}, \frac{w}{y}\right\} \Rightarrow \Delta, \phi\left\{\frac{u}{x}, \frac{w}{y}\right\}}{\Gamma, (TC_{x,y}\varphi)(s,t) \Rightarrow \Delta, \phi\left\{\frac{s}{x}, \frac{t}{y}\right\}} \tag{18}$$

where x, y should not occur free in $\Gamma \cup \Delta$, and u, v, w should not occurr free in Γ, Δ, ϕ and φ.

In what follows, we denote the sequent $\psi\left\{\frac{u}{x}, \frac{v}{y}\right\}, \psi\left\{\frac{v}{x}, \frac{w}{y}\right\} \Rightarrow \psi\left\{\frac{u}{x}, \frac{w}{y}\right\}$ by $Trans_{x,y}[\psi]$. The next theorem proves that TC_G is more adequate for ancestral logic than TC'_G.

Theorem 1. *TC_G is an extension TC'_G and all the sequents from Proposition 3 are provable in it.*

Proof. (Outline) In TC_G Rule (15) is derivable by taking for ϕ in Rule (18) the formula $\psi(x) \to \psi\left\{\frac{y}{x}\right\}$, for which $Trans_{x,y}[\phi]$ is clearly provable. To show that the first sequent in (16) is provable in TC_G, take for ϕ in Rule (18) the formula $\varphi(x,y) \vee \exists z\left((TC_{x,y}\varphi)(x,z) \wedge \varphi(z,y)\right)$. The provability of the other sequents from Proposition 3 then easily follows. □

Proposition 4. *In TC_G all the TC-counterparts of the rules in Proposition 2 are derivable.*

Since each of the two forms of the transitive closure operator can be expressed in terms of the other, it is interesting to explore the connection between RTC_G and

TC_G. Let φ be a formula in $\mathcal{L}_{TC}$. Define φ^* to be its $\mathcal{L}_{RTC}$-translation by induction as follows: for each formula φ in first-order language define $\varphi^* := \varphi$, and define $((TC_{x,y}A)(s,t))^*$ to be the formula: $\exists z \left(A^* \left\{\frac{s}{x}, \frac{z}{y}\right\} \wedge (RTC_{x,y}A^*)(z,t)\right)$. Let ψ be a formula in $\mathcal{L}_{RTC}$. Then ψ' is the formula in $\mathcal{L}_{TC}$ defined by induction as follows: for each formula ψ in first-order language define $\psi' := \psi$, and define $((RTC_{x,y}A)(s,t))'$ to be the formula $(TC_{x,y}A')(s,t) \vee s = t$. We use the standard abbreviations: Γ^* for $\{\varphi^* | \varphi \in \Gamma\}$ and Γ' for $\{\varphi' | \varphi \in \Gamma\}$.

First we show that any theorem of TC_G can be translated into a theorem of RTC_G, and vice versa.

Proposition 5. *The following holds:*

1. $\vdash_{TC_G} \Gamma \Rightarrow \Delta$ *implies* $\vdash_{RTC_G} \Gamma^* \Rightarrow \Delta^*$.
2. $\vdash_{RTC_G} \Gamma \Rightarrow \Delta$ *implies* $\vdash_{TC_G} \Gamma' \Rightarrow \Delta'$.

Note that neither $(\varphi')^*$ nor $(\varphi^*)'$ is syntactically equal to φ. For instance, for $\varphi = (TC_{x,y}P(x,y))(s,t)$, $(\varphi^*)'$ is $\exists z (P(s,z) \wedge ((TC_{x,y}P(x,y))(z,t) \vee z = t))$. However, as the next proposition will show, $(\varphi')^*$ and $(\varphi^*)'$ are provably equivalent to φ.

Proposition 6. *The following holds:*

1. $\vdash_{TC_G} (\varphi^*)' \Rightarrow \varphi$ *and* $\vdash_{TC_G} \varphi \Rightarrow (\varphi^*)'$.
2. $\vdash_{RTC_G} (\varphi')^* \Rightarrow \varphi$ *and* $\vdash_{RTC_G} \varphi \Rightarrow (\varphi')^*$.

Theorem 2. *TC_G and RTC_G are equivalent, i.e. the following holds:*

1. $\vdash_{RTC_G} \Gamma \Rightarrow \Delta$ *iff* $\vdash_{RTC_G} \Gamma' \Rightarrow \Delta'$.
2. $\vdash_{TC_G} \Gamma \Rightarrow \Delta$ *iff* $\vdash_{RTC_G} \Gamma^* \Rightarrow \Delta^*$.

Proof. Follows immediately from Propositions 5 and 6. □

Next we explore some proof-theoretical properties of the system TC_G. A system is said to be consistent if it does not admit a proof of the absurd, i.e. the empty sequent. In $\mathcal{LK}_=$, as well as in TC_G, formulas never disappear, except in cuts (the only other simplification allowed is contraction, in which a repetition is reduced). From this follows that there can be no cut-free proof of the empty sequent. Thus, by proving a weak version of the cut elimination theorem which states cut admissibility only for proofs ending with the empty sequent, one establishes the consistency of the system.

In [9] Gentzen proved the consistency of PA_G (Gentzen-style system for PA)[5] by providing a constructive method for transforming any proof of the empty sequent into a cut-free proof. A crucial step in the proof is the elimination of all appearances of PA_G's induction rule from the end-piece of the proof.[6] First,

[5] It should be noted that Gentzen did not prove full cut elimination for PA_G, only consistency.

[6] The end-piece of a proof consists of all the sequents of the proof encountered if we ascend each path starting from the end-sequent and stop when we arrive to an operational inference rule. Thus the lower sequent of this inference rule belongs to the end-piece, but its upper sequents do not.

all free variables which are not used as eigenvariables in the end-piece of the proof are replaced by constants. Then, any application of the induction rule up to a specific natural number is replaced by a corresponding number of structural inference rules. The transformation is done in the following way. Assume that the following application of PA_G's induction rule appears within an end-piece

$$\dfrac{\begin{array}{c}\vdots\, P\\ \psi\left\{\frac{a}{x}\right\} \Rightarrow \psi\left\{\frac{s(a)}{x}\right\}\end{array}}{\psi\left\{\frac{0}{x}\right\} \Rightarrow \psi\left\{\frac{t}{x}\right\}}$$

where P denotes the sub-proof ending with the sequent $\psi\left\{\frac{a}{x}\right\} \Rightarrow \psi\left\{\frac{s(a)}{x}\right\}$. Since all free variables were eliminated, t is a closed term and hence there is a term $s(\ldots(s(0))$ such that $\Rightarrow s(\ldots(s(0)) = t$ is provable in PA_G without essential cuts or induction. Therefore, there is also a proof of $\psi(s(\ldots(s(0))) \Rightarrow \psi(t)$ without essential cuts or induction. Let $P(b)$ be the proof obtained from P by replacing a by b throughout the proof. Replace any occurrence of the induction rule by

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P(0)\\ \psi\left\{\frac{0}{x}\right\} \Rightarrow \psi\left\{\frac{s(0)}{x}\right\}\end{array} \quad \begin{array}{c}\vdots\, P(s(0))\\ \psi\left\{\frac{s(0)}{x}\right\} \Rightarrow \psi\left\{\frac{s(s(0))}{x}\right\}\end{array}}{\psi\left\{\frac{0}{x}\right\} \Rightarrow \psi\left\{\frac{s(s(0))}{x}\right\}} \quad \begin{array}{c}\vdots\, P(s(s(0)))\\ \psi\left\{\frac{s(s(0))}{x}\right\} \Rightarrow \psi\left\{\frac{s(s(s(0)))}{x}\right\}\end{array}}{\psi\left\{\frac{0}{x}\right\} \Rightarrow \psi\left\{\frac{s(s(s(0)))}{x}\right\}}$$

These consecutive cuts are carried on up to the sequent $\psi\left\{\frac{0}{x}\right\} \Rightarrow \psi\left\{\frac{s(\ldots(s(0))}{x}\right\}$. Then one more cut is used on the sequent $\psi(s(\ldots(s(0))) \Rightarrow \psi(t)$ to obtain a proof of $\psi\left\{\frac{0}{x}\right\} \Rightarrow \psi\left\{\frac{t}{x}\right\}$.

Can a similar method be applied to the TC-induction rule? The problem is that Gentzen's transformation of the induction rule uses special features of the natural numbers that generally do not exist in TC_G. To see this, notice that the induction rule (Rule (18)) entails all instances of PA_G's induction rule by taking φ to be $s(x) = y$ and ϕ to be $\psi(x) \to \psi\left\{\frac{y}{x}\right\}$. However, in the general case φ is an arbitrary formula. Thus, unlike in PA_G, we do not have a "built in" measure for the φ-distance between two arbitrary closed terms s and t. The φ-path from s to t is not known apriori. Moreover, it does not have to be unique.

Unfortunately, this generalization of the induction principle renders this standard method for analyzing PA_G inapplicable. Thus, one should look for useful fragments of TC_G in which cuts can be eliminated from proofs of the empty sequent. One such fragment can be obtained via restricting TC_G's induction rule by allowing only φ's of the form $y = t$, where x is the only free variable in t. In this way we force a deterministic φ-path between any two closed terms, while keeping the system strong enough for the task of mechanizing mathematics, as its restricted induction rule still includes that of PA_G. Exploring this direction will be left for further research.

Another proof-theoretical method which arises from Gentzen's constructive consistency proofs is the assignment of ordinals to proof systems. In Gentzen's method, each system is assigned the least ordinal number needed for its constructive consistency proof. This provides a measure for a complexity of a system which is useful for comparing different proof systems. The constructive consistency proof of PA_G entails that the ordinal number of PA_G is at most ε_0, and another theorem of Gentzen [10] shows that it is exactly ε_0.

Definition 6. *The system TC_A is obtained by augmenting TC_G with the standard axioms for successor, addition, and multiplication, together with the axiom characterizing the natural numbers in ancestral logic (Axiom (1)).*

Proposition 7. *TC_A is equivalent to PA_G.*

Proof. (Outline) TC_A is an extension of PA_G, since Rule (18) entails all instances of PA_G's induction rule. In [17] it was shown how it is possible, using a β-function, to encode in PA_G finite sequences and thus define the TC operator. It is easy to see that the system TC_A is equivalent to PA_G, in the sense that there are provability preserving translation algorithms between them. □

Corollary 1. *The ordinal number of the system TC_A is ε_0.*

4 Conclusions and Further Research

In this paper we reviewed the expressive power of logics augmented by a transitive closure operator and explored their reasoning potential. This work focused on working out this potential by presenting effective sound proof systems for ancestral logic that are strong enough for various mathematical needs. The next goal is to improve the computational efficiency of these systems, in order to make them suitable for mechanization.

We believe that ancestral logic should suffice for most of applicable mathematics. Substantiating this claim by creating formal systems based on ancestral logic and formalizing in them large portions of mathematics, is a further future work. A promising candidate for serving as the basis for such system is the predicative set theory PZF, presented in [3,4], which resembles ZF and is suitable for mechanization. The key elements of PZF are that it uses syntactic safety relations between formulas and sets of variables, and that its underlying logic is ancestral logic, which makes it possible to provide inductive definitions of relations and functions. An important criterion for the adequacy of ancestral logic for the task of formalizing mathematics is the extent to which such formalization can be done in a natural way, as close as possible to real mathematical practice.

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Appendix

In what follows, for readability, we shall not distinguish between the sequents $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$ and $\varphi, \psi, \Gamma \Rightarrow \Delta$, or $\Gamma \Rightarrow \Delta, \varphi \vee \psi$ and $\Gamma \Rightarrow \Delta, \varphi, \psi$, as they are provable from one another.

Proof of Proposition 2:

– The first rule in (6):

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \varphi\left\{\frac{s}{x}, \frac{r}{y}\right\}}{\Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(s,r)}\ (3) \qquad \Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(r,t)}{\Gamma \Rightarrow \Delta, (RTC_{x,y}\varphi)(s,t)}\ (4)$$

The proof of the second rule in (6) is analogous.

- The first rule in (7): Consider the following proof, P_1:

$$\dfrac{\dfrac{\dfrac{\Rightarrow (RTC_{x,y}\varphi)(y,y)}{s=y \Rightarrow (RTC_{x,y}\varphi)(s,y)} \quad \varphi\left\{\frac{y}{x},\frac{z}{y}\right\} \Rightarrow \varphi\left\{\frac{y}{x},\frac{z}{y}\right\}}{s=y,\varphi\left\{\frac{y}{x},\frac{z}{y}\right\} \Rightarrow (RTC_{x,y}\varphi)(s,y) \wedge \varphi\left\{\frac{y}{x},\frac{z}{y}\right\}}}{s=y,\varphi\left\{\frac{y}{x},\frac{z}{y}\right\} \Rightarrow \exists w\left((RTC_{x,y}\varphi)(s,w) \wedge \varphi\left\{\frac{w}{x},\frac{z}{y}\right\}\right)}$$

  The sequent $(RTC_{x,y}\varphi)(s,w), \varphi\left\{\frac{w}{x}\right\} \Rightarrow (RTC_{x,y}\varphi)(s,y)$ is provable in RTC_G using (6). Thus, by applying standard $\mathcal{LK}_=$ rules we can construct a proof, P_2, of $\exists w\left((RTC_{x,y}\varphi)(s,w) \wedge \varphi\left\{\frac{w}{x}\right\}\right), \varphi\left\{\frac{y}{x},\frac{z}{y}\right\} \Rightarrow \exists w\left((RTC_{x,y}\varphi)(s,w) \wedge \varphi\left\{\frac{w}{x},\frac{z}{y}\right\}\right)$. Denote by $A(y)$ the formula $\exists w\left((RTC_{x,y}\varphi)(s,w) \wedge \varphi\left\{\frac{w}{x}\right\}\right) \vee s=y$. From P_1 and P_2 we obtain a proof of the sequent $A(y), \varphi\left\{\frac{y}{x},\frac{z}{y}\right\} \Rightarrow A\left\{\frac{z}{y}\right\}$, from which, using Rule (5), we deduce $A\left\{\frac{s}{y}\right\}, (RTC_{x,y}\varphi)(s,t) \Rightarrow A\left\{\frac{t}{y}\right\}$. Since $\Rightarrow A\left\{\frac{s}{y}\right\}$ is derivable from the equality axiom, applying a cut on it results in the desired end-sequent. The proof of the second rule in (7) is symmetric.

- The left rule in (8): The sequent $\varphi(x,y), (RTC_{y,x}\varphi)(x,s) \Rightarrow (RTC_{y,x}\varphi)(y,s)$ is provable in RTC_G using (6). Thus, we can construct the following proof:

$$\dfrac{\dfrac{\dfrac{\varphi\left\{\frac{z}{y},\frac{s}{x}\right\} \Rightarrow \varphi\left\{\frac{z}{y},\frac{s}{x}\right\}}{\varphi\left\{\frac{z}{y},\frac{s}{x}\right\} \Rightarrow (RTC_{y,x}\varphi)(z,s)}\,(3) \quad \dfrac{\varphi(x,y), (RTC_{y,x}\varphi)(x,s) \Rightarrow (RTC_{y,x}\varphi)(y,s)}{(RTC_{x,y}\varphi)(z,t), (RTC_{y,x}\varphi)(z,s) \Rightarrow (RTC_{y,x}\varphi)(t,s)}\,(5)}{\varphi\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge (RTC_{x,y}\varphi)(z,t) \Rightarrow (RTC_{y,x}\varphi)(t,s)}}{\exists z\left(\varphi\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge (RTC_{x,y}\varphi)(z,t)\right) \Rightarrow (RTC_{y,x}\varphi)(t,s)}$$

  The sequent $(RTC_{x,y}\varphi)(s,t) \Rightarrow s=t, \exists z\left(\varphi\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge (RTC_{x,y}\varphi)(z,t)\right)$ is provable in RTC_G using Rule (7) and $s=t \Rightarrow (RTC_{y,x}\varphi)(t,s)$ is provable using Axiom (2). From this, by cuts, we obtain a proof of $(RTC_{x,y}\varphi)(s,t) \Rightarrow (RTC_{y,x}\varphi)(t,s)$. The proof of the right rule is symmetric.

- The left rule in (9): In RTC_G the sequent $s=t \Rightarrow \left(RTC_{u,v}\varphi\left\{\frac{u}{x},\frac{v}{y}\right\}\right)(s,t)$ is provable. By a method similar to the one used in the proof of (8) we get the provability of $\exists z\left((RTC_{x,y}\varphi)(s,z) \wedge \varphi\left\{\frac{z}{x},\frac{t}{y}\right\}\right) \Rightarrow \left(RTC_{u,v}\varphi\left\{\frac{u}{x},\frac{v}{y}\right\}\right)(s,t)$. Applying cuts and Rule (7) results in a proof of the sequent $(RTC_{x,y}\varphi)(s,t) \Rightarrow \left(RTC_{u,v}\varphi\left\{\frac{u}{x},\frac{v}{y}\right\}\right)(s,t)$. The proof of the right rule is symmetric.

- Rule (10): Consider the following two proofs:

$$\dfrac{\dfrac{\varphi \Rightarrow \psi}{\varphi\left\{\frac{s}{x},\frac{z}{y}\right\} \Rightarrow \psi\left\{\frac{s}{x},\frac{z}{y}\right\}} \quad \Rightarrow (RTC_{x,y}\psi)(z,z)}{\varphi\left\{\frac{s}{x},\frac{z}{y}\right\} \Rightarrow (RTC_{x,y}\psi)(s,z)}\,(6)$$

$$\dfrac{(RTC_{x,y}\psi)(s,z) \Rightarrow (RTC_{x,y}\psi)(s,z) \qquad \dfrac{\varphi \Rightarrow \psi}{\varphi\left\{\frac{z}{x},\frac{u}{y}\right\} \Rightarrow \psi\left\{\frac{z}{x},\frac{u}{y}\right\}}}{\dfrac{(RTC_{x,y}\psi)(s,z), \varphi\left\{\frac{z}{x},\frac{u}{y}\right\} \Rightarrow (RTC_{x,y}\psi)(s,u)}{(RTC_{x,y}\psi)(s,z), (RTC_{x,y}\varphi)(z,t) \Rightarrow (RTC_{x,y}\psi)(s,t)}\ (5)}\ (6)$$

From the above proofs we can deduce $\exists z\left(\varphi\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge (RTC_{x,y}\varphi)(z,t)\right) \Rightarrow (RTC_{x,y}\psi)(s,t)$. Clearly, the sequent $s = t \Rightarrow (RTC_{y,x}\psi)(s,t)$ is provable in RTC_G using Axiom (2). Using Rule (7) we get $(RTC_{x,y}\varphi)(s,t) \Rightarrow s = t, \exists z\left(\varphi\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge (RTC_{x,y}\varphi)(z,t)\right)$, and two cuts result in a proof of $(RTC_{x,y}\varphi)(s,t) \Rightarrow (RTC_{x,y}\psi)(s,t)$.

- Rule (11): Rule (4) entails the existence of a proof in RTC_G of the sequent $(RTC_{x,y}\varphi)(s,u), (RTC_{x,y}\varphi)(u,v) \Rightarrow (RTC_{x,y}\varphi)(s,v)$. By Rule (5) we get a proof of $(RTC_{x,y}\varphi)(s,s), (RTC_{u,v}(RTC_{x,y}\varphi)(u,v))(s,t) \Rightarrow (RTC_{x,y}\varphi)(s,t)$. A cut on the axiom $\Rightarrow (RTC_{x,y}\varphi)(s,s)$ results in the desired proof.
- The left rule in (12): From $\varphi\left\{\frac{s}{x}\right\} \Rightarrow$, by standard $\mathcal{LK}_=$ rules, we can derive $\exists z\left(\varphi\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge (RTC_{x,y}\varphi)(z,t)\right) \Rightarrow$. By Rule (7) we have $(RTC_{x,y}\varphi)(s,t) \Rightarrow s = t, \exists z\left(\varphi\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge (RTC_{x,y}\varphi)(z,t)\right)$. Thus, a cut results in a proof of $(RTC_{x,y}\varphi)(s,t) \Rightarrow s = t$. The proof of the right rule in (12) is analogous.

□

Proof of Theorem 1:
Clearly $Trans_{x,y}[\psi(x) \to \psi\left\{\frac{y}{x}\right\}]$ is provable. Thus, we derive Rule (15) by:

$$\dfrac{\dfrac{\dfrac{\psi(x), \varphi(x,y) \Rightarrow \psi\left\{\frac{y}{x}\right\}}{\varphi(x,y) \Rightarrow \psi(x) \to \psi\left\{\frac{y}{x}\right\}} \quad Trans_{x,y}[\psi(x) \to \psi\left\{\frac{y}{x}\right\}]}{(TC_{x,y}\varphi)(s,t) \Rightarrow \psi\left\{\frac{s}{x}\right\} \to \psi\left\{\frac{t}{x}\right\}}\ (18)}{\psi\left\{\frac{s}{x}\right\}, (TC_{x,y}\varphi)(s,t) \Rightarrow \psi\left\{\frac{t}{x}\right\}}$$

To see that the first sequent in (16) is provable in TC_G, take ϕ to be $\varphi(x,y) \vee \exists z((TC_{x,y}\varphi)(x,z) \wedge \varphi(z,y))$. For any two terms r_1, r_2, denote by A_{r_1,r_2} the formula $\exists z((TC_{x,y}\varphi)(r_1,z) \wedge \varphi(z,r_2))$. Clearly, $\varphi(x,y) \Rightarrow \varphi(x,y) \vee A_{x,y}$ is provable in TC_G. We show that $Trans_{x,y}[\varphi(x,y) \vee A_{x,y}]$ is also provable. Observe the following sub-proof:

$$\dfrac{\dfrac{(TC_{x,y}\varphi)(u,v), (TC_{x,y}\varphi)(v,a) \Rightarrow (TC_{x,y}\varphi)(u,a) \quad (TC_{x,y}\varphi)(u,a), \varphi(a,w) \Rightarrow A_{u,w}}{(TC_{x,y}\varphi)(u,v), (TC_{x,y}\varphi)(v,a) \wedge \varphi(a,w) \Rightarrow A_{u,w}}}{(TC_{x,y}\varphi)(u,v), A_{v,w} \Rightarrow A_{u,w}}$$

It is easy to see that the sequent $(TC_{x,y}\varphi)(u,v), \varphi(v,w) \Rightarrow A_{u,w}$ is provable in TC_G, so we can obtain a proof of the sequent $(TC_{x,y}\varphi)(u,v), \varphi(v,w) \vee A_{v,w} \Rightarrow \varphi(u,w) \vee A_{u,w}$. The sequent $\varphi(u,v) \vee A_{u,v} \Rightarrow (TC_{x,y}\varphi)(u,v)$ is also provable in TC_G, hence, using a cut we get a proof of $\phi(u,v), \phi(v,w) \Rightarrow \phi(u,w)$. Now we can construct the following derivation:

$$\dfrac{\varphi(x,y) \Rightarrow \varphi(x,y) \vee \exists z((TC_{x,y}\varphi)(x,z) \wedge \varphi(z,y)) \quad Trans_{x,y}[\phi]}{(TC_{x,y}\varphi)(s,t) \Rightarrow \varphi\left\{\frac{s}{x},\frac{t}{y}\right\}, \exists z((TC_{x,y}\varphi)(s,z) \wedge \varphi(z,t))}\ (18)$$

The proof of the second sequent in (16) is similar. To see that the sequents in (17) are provable, notice that both $\varphi\left\{\frac{s}{x},\frac{t}{y}\right\}\vee\exists z\left(\left(TC_{x,y}\varphi\right)(s,z)\wedge\varphi(z,t)\right)\Rightarrow\varphi\left\{\frac{t}{y}\right\}$ and $\varphi\left\{\frac{s}{x},\frac{t}{y}\right\}\vee\exists w\left(\varphi(s,z)\wedge\left(TC_{x,y}\varphi\right)(z,t)\right)\Rightarrow\varphi\left\{\frac{s}{x}\right\}$ are provable in TC_G. From this, using (16) and cuts, we obtain the desired proofs. $\square$

Proof of Proposition 5:

Lemma 1. *The following holds:*

- $\left(\varphi\left\{\frac{s}{x},\frac{t}{y}\right\}\right)^{*}=\varphi^{*}\left\{\frac{s}{x},\frac{t}{y}\right\}$ and $\left(\varphi\left\{\frac{s}{x},\frac{t}{y}\right\}\right)'=\varphi'\left\{\frac{s}{x},\frac{t}{y}\right\}$.
- $(\neg\varphi)^{*}=\neg\varphi^{*}$ *and* $(\neg\varphi)'=\neg\varphi'$.
- $(\varphi\circ\psi)^{*}=\varphi^{*}\circ\psi^{*}$ *and* $(\varphi\circ\psi)'=\varphi'\circ\psi'$, where $\circ\in\{\wedge,\vee,\rightarrow\}$.
- $(Qx\varphi)^{*}=Qx\varphi^{*}$ *and* $(Qx\varphi)'=Qx\varphi'$, where $Q\in\{\forall,\exists\}$.

The proofs of (1) and (2) are carried out by induction, we state here only the cases concerning the TC and RTC operators.

- Rule (13): By standard $\mathcal{LK}_{=}$ rules derive from $\Rightarrow\varphi^{*}\left\{\frac{s}{x},\frac{t}{y}\right\}$ and the axiom $\Rightarrow\left(RTC_{x,y}\varphi^{*}\right)(t,t)$ the sequent $\Rightarrow\exists z\left(\varphi^{*}\left\{\frac{s}{x},\frac{z}{y}\right\}\wedge\left(RTC_{x,y}\varphi^{*}\right)(z,t)\right)$.
- Rule (14): Rule (6) entails the existence of a proof in RTC_G of the sequent $\exists z\left(\varphi^{*}\left\{\frac{r}{x},\frac{z}{y}\right\}\wedge\left(RTC_{x,y}\varphi^{*}\right)(z,t)\right)\Rightarrow\left(RTC_{x,y}\varphi^{*}\right)(r,t)$. A cut on the hypothesis $\Rightarrow\exists z\left(\varphi^{*}\left\{\frac{r}{x},\frac{z}{y}\right\}\wedge\left(RTC_{x,y}\varphi^{*}\right)(z,t)\right)$ results in a proof of the sequent $\Rightarrow\left(RTC_{x,y}\varphi^{*}\right)(r,t)$. Applying Rule (4) on $\Rightarrow\left(RTC_{x,y}\varphi^{*}\right)(r,t)$ and the axiom $\left(RTC_{x,y}\varphi^{*}\right)(z,r)\Rightarrow\left(RTC_{x,y}\varphi^{*}\right)(z,r)$ results in a proof of $\left(RTC_{x,y}\varphi^{*}\right)(z,r)\Rightarrow\left(RTC_{x,y}\varphi^{*}\right)(z,t)$. By standard $\mathcal{LK}_{=}$ rules we derive $\exists z\left(\varphi^{*}\left\{\frac{s}{x},\frac{z}{y}\right\}\wedge\left(RTC_{x,y}\varphi^{*}\right)(z,r)\right)\Rightarrow\exists z\left(\varphi^{*}\left\{\frac{s}{x},\frac{z}{y}\right\}\wedge\left(RTC_{x,y}\varphi^{*}\right)(z,t)\right)$. The desired sequent is now obtained by one more cut on the second hypothesis $\Rightarrow\exists z\left(\varphi^{*}\left\{\frac{s}{x},\frac{z}{y}\right\}\wedge\left(RTC_{x,y}\varphi^{*}\right)(z,r)\right)$.
- Rule (18): From $Trans_{x,y}\left[\phi^{*}\right]$ we can deduce $\phi^{*}(s,x),\phi^{*}(x,y)\Rightarrow\phi^{*}(s,y)$. Using a cut on $\varphi^{*}(x,y)\Rightarrow\phi^{*}(x,y)$ we get $\phi^{*}(s,x),\varphi^{*}(x,y)\Rightarrow\phi^{*}(s,y)$. Applying Rule (5) results in $\phi^{*}(s,z),\left(RTC_{x,y}\varphi^{*}\right)(z,t)\Rightarrow\phi^{*}(s,t)$. Using a cut on $\varphi^{*}(s,z)\Rightarrow\phi^{*}(s,z)$ we get the sequent $\varphi^{*}(s,z),\left(RTC_{x,y}\varphi^{*}\right)(z,t)\Rightarrow\phi^{*}(s,t)$, from which $\exists z\left(\varphi^{*}(s,z)\wedge\left(RTC_{x,y}\varphi^{*}\right)(z,t)\right)\Rightarrow\phi^{*}(s,t)$ is easily derivable.
- Axiom (2): The translation of the axiom is $\Rightarrow\left(TC_{x,y}\varphi'\right)(s,s)\vee s=s$, which is easily derivable from the equality axioms.
- Rule (3): Using Rule and introduction of $\vee$ on the right we can deduce $\Rightarrow\left(TC_{x,y}\varphi'\right)(s,t)\vee s=t$ from $\Rightarrow\varphi'\left\{\frac{s}{x},\frac{t}{y}\right\}$.
- Rule (4): It is easy to see that from the sequents $\Rightarrow\left(TC_{x,y}\varphi'\right)(s,r),s=r$ and $\Rightarrow\left(TC_{x,y}\varphi'\right)(r,t),r=t$ we can prove $\Rightarrow\left(TC_{x,y}\varphi'\right)(s,t),s=t$ using Rule (14) and equality rules.

– Rule (5): As Rule (15) is derivable in TC_G, an application of Rule (5) can be transformed into the following derivation:

$$\dfrac{\dfrac{\psi'(x), \varphi'(x,y) \Rightarrow \psi'\left\{\frac{y}{x}\right\}}{\psi'\left\{\frac{s}{x}\right\}, (TC_{x,y}\varphi')(s,t) \Rightarrow \psi'\left\{\frac{t}{x}\right\}}\,(15) \qquad \psi'\left\{\frac{s}{x}\right\}, s=t \Rightarrow \psi'\left\{\frac{t}{x}\right\}}{\psi'\left\{\frac{s}{x}\right\}, (TC_{x,y}\varphi')(s,t) \vee s=t \Rightarrow \psi'\left\{\frac{t}{x}\right\}}$$

□

Proof of Proposition 6:

If φ does not contain the TC or RTC operator, then $(\varphi')^*$ and $(\varphi^*)'$ are syntactically equal to φ, hence provably equivalent to it.

For (1) assume that $\varphi := (RTC_{x,y}A)(s,t)$. By the induction hypothesis we have $\vdash_{RTC_G} (A')^* \Rightarrow A$, thus by (10) the sequent $\left(RTC_{x,y}(A')^*\right)(s,t) \Rightarrow (RTC_{x,y}A)(s,t)$ is also provable in RTC_G. It is easy to check that the sequent $\exists z\left((A')^*\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge RTC_{x,y}(A')^*(z,t)\right) \vee s=t \Rightarrow \left(RTC_{x,y}(A')^*\right)(s,t)$ is provable in RTC_G (using (6) and (2)). A cut on the last two sequents results in a proof of $\exists z\left((A')^*\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge RTC_{x,y}(A')^*(z,t)\right) \vee s=t \Rightarrow (RTC_{x,y}A)(s,t)$. For the converse, denote $\exists z\left((A')^*\left\{\frac{u}{x},\frac{z}{y}\right\} \wedge RTC_{x,y}(A')^*(z,w)\right) \vee s=t$ by ψ (notice that $(\varphi')^*$ is $\psi\left\{\frac{s}{u},\frac{t}{w}\right\}$). It is easy to see that $\psi\left\{\frac{s}{u},\frac{x}{w}\right\}, (A')^* \Rightarrow \psi\left\{\frac{s}{u},\frac{y}{w}\right\}$ is provable in RTC_G. Applying Rule (5) results in a proof of the sequent $\psi\left\{\frac{s}{u},\frac{s}{w}\right\}, \left(RTC_{x,y}(A')^*\right)(s,t) \Rightarrow \psi\left\{\frac{s}{u},\frac{t}{w}\right\}$. The sequent $\Rightarrow \psi\left\{\frac{s}{u},\frac{s}{w}\right\}$ is clearly provable using the equality axiom, thus, a cut entails a proof of the sequent $\left(RTC_{x,y}(A')^*\right)(s,t) \Rightarrow (\varphi')^*$. As before, by the induction hypothesis we have that $\vdash_{RTC_G} A \Rightarrow (A')^*$, so by (10) the sequent $(RTC_{x,y}A)(s,t) \Rightarrow \left(RTC_{x,y}(A')^*\right)(s,t)$ is also provable in RTC_G, and by one cut we obtain a proof of $(RTC_{x,y}A)(s,t) \Rightarrow (\varphi')^*$.

For (2) assume that $\varphi := (TC_{x,y}A)(s,t)$. It is easy to check that the sequent $\exists z\left((A^*)'\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge \left(TC_{x,y}(A^*)'(z,t) \vee z=t\right)\right) \Rightarrow \left(TC_{x,y}(A^*)'\right)(s,t)$ is provable in TC_G. By the induction hypothesis we have that $\vdash_{TC_G} (A^*)' \Rightarrow A$, so by the TC-counterpart of (10) the sequent $\left(TC_{x,y}(A^*)'\right)(s,t) \Rightarrow (TC_{x,y}A)(s,t)$ is also provable in TC_G. Now, applying a cut results in a proof of the sequent $\exists z\left((A^*)'\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge \left(TC_{x,y}(A^*)'(z,t) \vee z=t\right)\right) \Rightarrow (TC_{x,y}A)(s,t)$. For the converse, notice that the derivability of (16) in TC_G entails the provability of $\left(TC_{x,y}(A^*)'\right)(s,t) \Rightarrow (A^*)'\left\{\frac{s}{x},\frac{t}{y}\right\} \vee \exists z\left((A^*)'\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge \left(TC_{x,y}(A^*)'\right)(z,t)\right)$. Clearly, the sequent $(A^*)'\left\{\frac{s}{x},\frac{t}{y}\right\} \Rightarrow \exists z\left((A^*)'\left\{\frac{s}{x},\frac{z}{y}\right\} \wedge z=t\right)$ is provable in TC_G, and again, using the induction hypothesis on A together with the TC-counterpart of (10) we get that $(TC_{x,y}A)(s,t) \Rightarrow \left(TC_{x,y}(A^*)'\right)(s,t)$ is provable in TC_G. Applying cuts results in a proof of the sequent $(TC_{x,y}A)(s,t) \Rightarrow (\varphi^*)'$.

□